

# Groups with Boundedly Finite Conjugacy Classes of Commutators

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**ABSTRACT.** In 1954 B. H. Neumann discovered that if  $G$  is a group in which all conjugacy classes are finite with bounded size, then the derived group  $G'$  is finite. Later (in 1957) Wiegold found an explicit bound for the order of  $G'$ . We study groups in which the conjugacy classes containing commutators are finite with bounded size. We obtain the following results.

Let  $G$  be a group and  $n$  a positive integer.

If  $|x^G| \leq n$  for any commutator  $x \in G$ , then the second derived group  $G''$  is finite with  $n$ -bounded order.

If  $|x^{G'}| \leq n$  for any commutator  $x \in G$ , then the order of  $\gamma_3(G')$  is finite and  $n$ -bounded.

## 1. Introduction

Given a group  $G$  and an element  $x \in G$ , we write  $x^G$  for the conjugacy class containing  $x$ . Of course, if the number of elements in  $x^G$  is finite, we have  $|x^G| = [G : C_G(x)]$ . A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. One of B. H. Neumann's discoveries was that in a BFC-group the derived group  $G'$  is finite [3]. It follows that if  $|x^G| \leq n$  for each  $x \in G$ , then the order of  $G'$  is bounded by a number depending only on  $n$ . A first explicit bound for the order of  $G'$  was found by J. Wiegold [7], and the best known was obtained in [1] (see also [4] and [6]).

In the present article we deal with groups  $G$  such that  $|x^G| \leq n$  whenever  $x$  is a commutator, that is,  $x = [x_1, x_2]$  for suitable  $x_1, x_2 \in G$ . Here and throughout the article we write  $[x_1, x_2]$  for  $x_1^{-1}x_2^{-1}x_1x_2$ . As usual, we denote by  $G'$  the derived group of  $G$  and by  $G''$  the derived group of  $G'$  (the second derived group of  $G$ ).

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**THEOREM 1.1.** *Let  $n$  be a positive integer and  $G$  a group in which  $|x^G| \leq n$  for any commutator  $x$ . Then  $|G''|$  is finite and  $n$ -bounded.*

Further, we consider groups  $G$  in which  $|x^{G'}| \leq n$  whenever  $x$  is a commutator.

**THEOREM 1.2.** *Let  $n$  be a positive integer and  $G$  a group in which  $|x^{G'}| \leq n$  for any commutator  $x$ . Then  $|\gamma_3(G')|$  is finite and  $n$ -bounded.*

Here  $\gamma_3(G')$  denotes the third term of the lower central series of  $G'$ . We do not know whether under hypothesis of Theorem 1.2 the second derived group  $G''$  must necessarily be finite. Note that under hypothesis of Theorem 1.1  $\gamma_3(G)$  can be infinite. This can be shown using any example of an infinite torsion-free metabelian group whose commutator quotient is finite (see for instance [2]).

## 2. Proofs

Let  $G$  be a group generated by a set  $X$  such that  $X = X^{-1}$ . Given an element  $g \in G$ , we write  $l_X(g)$  for the minimal number  $l$  with the property that  $g$  can be written as a product of  $l$  elements of  $X$ . Clearly,  $l_X(g) = 0$  if and only if  $g = 1$ . We call  $l_X(g)$  the length of  $g$  with respect to  $X$ .

**LEMMA 2.1.** *Let  $H$  be a group generated by a set  $X = X^{-1}$  and let  $K$  be a subgroup of finite index  $m$  in  $H$ . Then each coset  $Kb$  contains an element  $g$  such that  $l_X(g) \leq m - 1$ .*

**PROOF.** If  $b \in K$ , the result is obvious. Therefore we assume that  $b \notin K$ . Choose  $g \in Kb$  in such a way that  $s = l_X(g)$  is as small as possible and suppose that  $s \geq m$ . Write  $g = x_1 \cdots x_s$  with  $x_i \in X$  and set  $y_j = x_1 \cdots x_j$  for  $j = 1, \dots, s$ . Since  $s$  is the minimum of lengths of elements in  $Kb$ , it follows that none of the elements  $y_1, \dots, y_s$  lies in  $K$ . Thus, these  $s$  elements belong to the union of at most  $m - 1$  right cosets of  $K$  and we conclude that  $Ky_i = Ky_j$  for some  $1 \leq i < j \leq s$ . It is now easy to see that the element  $h = y_i x_{j+1} \cdots x_s$  belongs to  $Kb$  while  $l_X(h) < l_X(g)$ . This is a contradiction with the choice of  $g$ .  $\square$

In the sequel the above lemma will be used in the situation where  $H$  is the derived group of a group  $G$  and  $X$  is the set of commutators in  $G$ . Therefore we will write  $l(g)$  to denote the smallest number such that the element  $g \in G'$  can be written as a product of as many commutators. Recall that if  $H$  is a group and  $a \in H$ , the subgroup  $[H, a]$  is generated by all commutators of the form  $[h, a]$ , where  $h \in H$ . It is well-known that  $[H, a]$  is always normal in  $H$ . Recall that in any group  $G$  the following “standard commutator identities” hold.

- (1)  $[x, y]^{-1} = [y, x];$
- (2)  $[xy, z] = [x, z]^y [y, z];$
- (3)  $[x, yz] = [x, z][x, y]^z.$

In what follows the above identities will be used without explicit references.

We will now fix some notation and hypothesis.

**HYPOTHESIS 2.2.** *Let  $G$  be a group and  $K$  a subgroup containing  $H = G'$ . Let  $X$  denote the set of commutators in  $G$  and suppose that  $C_K(x)$  has finite index at most  $n$  in  $K$  for each  $x \in X$ . Let  $m$  be the maximum of indices of  $C_H(x)$  in  $H$ , where  $x \in X$ . Suppose further that  $a \in X$  and  $C_H(a)$  has index precisely  $m$  in  $H$ . Choose  $b_1, \dots, b_m \in H$  such that  $l(b_i) \leq m-1$  and  $a^H = \{a^{b_i}; i = 1, \dots, m\}$ . (The existence of such elements is guaranteed by Lemma 2.1.) Set  $U = C_K(\langle b_1, \dots, b_m \rangle)$ .*

**LEMMA 2.3.** *Assume Hypothesis 2.2. Then for any  $x \in X$  the subgroup  $[H, x]$  has finite  $m$ -bounded order.*

**PROOF.** Choose  $x \in X$ . Since  $C_H(x)$  has index at most  $m$  in  $H$ , by Lemma 2.1 we can choose elements  $y_1, \dots, y_m$  such that  $l(y_i) \leq m-1$  and  $[H, x]$  is generated by the commutators  $[y_i, x]$ . For each  $i = 1, \dots, m$  write  $y_i = y_{i1} \dots y_{i(m-1)}$ , where  $y_{ij} \in X$ . The standard commutator identities show that  $[y_i, x]$  can be written as a product of conjugates in  $H$  of the commutators  $[y_{ij}, x]$ . Let  $h_1, \dots, h_s$  be the conjugates in  $H$  of elements from the set  $\{x, y_{ij}; 1 \leq i, j \leq m\}$ . Since  $C_H(h)$  has finite index at most  $m$  in  $H$  for each  $h \in X$ , it follows that  $s$  is  $m$ -bounded. Let  $T = \langle h_1, \dots, h_s \rangle$ . It is clear that  $[H, x] \leq T'$  and so it is sufficient to show that  $T'$  has finite  $m$ -bounded order. Observe that  $C_H(h_i)$  has finite index at most  $m$  in  $H$  for each  $i = 1, \dots, s$ . It follows that the center  $Z(T)$  has index at most  $m^s$  in  $T$ . Thus, Schur's theorem [5, 10.1.4] tells us that  $T'$  has finite  $m$ -bounded order, as required.  $\square$

Note that the subgroup  $U$  has finite  $n$ -bounded index in  $K$ . This follows from the facts that  $l(b_i) \leq m-1$  and  $C_K(x)$  has index at most  $n$  in  $K$  for each  $x \in X$ .

The next lemma is somewhat analogous with Lemma 4.5 of Wiegold [7].

**LEMMA 2.4.** *Assume Hypothesis 2.2. Suppose that  $u \in U$  and  $ua \in X$ . Then  $[H, u] \leq [H, a]$ .*

**PROOF.** Since  $u \in U$ , it follows that  $(ua)^{b_i} = ua^{b_i}$  for each  $i = 1, \dots, m$ . Therefore the elements  $ua^{b_i}$  form the conjugacy class  $(ua)^H$ . For an arbitrary element  $g \in H$  there exists  $h \in \{b_1, \dots, b_m\}$  such that

$(ua)^g = ua^h$  and so  $u^g a^g = ua^h$ . Therefore  $[u, g] = a^h a^{-g} \in [H, a]$ . The lemma follows.  $\square$

**PROPOSITION 2.5.** *Assume Hypothesis 2.2 and write  $a = [d, e]$  for suitable  $d, e \in G$ . There exists a subgroup  $U_1 \leq U$  with the following properties.*

- (1) *The index of  $U_1$  in  $K$  is  $n$ -bounded;*
- (2)  *$[H, U_1] \leq [H, a]^{d^{-1}}$ ;*
- (3)  *$[H, [U_1, d]] \leq [H, a]$ .*

**PROOF.** Set

$$U_1 = U \cap U^{d^{-1}} \cap U^{d^{-1}e^{-1}}.$$

Since the index of  $U$  in  $K$  is  $n$ -bounded, we conclude that the index of  $U_1$  in  $K$  is  $n$ -bounded as well. Choose arbitrarily elements  $h_1, h_2 \in U_1$ . Write

$$[h_1 d, e h_2] = [h_1, h_2]^d [d, h_2] [h_1, e]^{dh_2} [d, e]^{h_2}$$

and so

$$[h_1 d, e h_2]^{h_2^{-1}} = [h_1, h_2]^{dh_2^{-1}} [d, h_2]^{h_2^{-1}} [h_1, e]^d [d, e].$$

Denote the product  $[h_1, h_2]^{dh_2^{-1}} [d, h_2]^{h_2^{-1}} [h_1, e]^d$  by  $u$ . Thus, the right hand side of the above equality is  $ua$  while, obviously, on the left hand side we have a commutator. Let us check that  $u \in U$ . We see that  $[h_1, h_2]^{dh_2^{-1}} \in U_1^{dh_2^{-1}} \leq U$  because  $U_1^d \leq U$ . By the same reason,  $[d, h_2]^{h_2^{-1}} \in U$ . Finally,  $[h_1, e]^d \in U_1^d U_1^{ed} \leq U$  so indeed  $u \in U$ . By Lemma 2.4,  $[H, u] \leq [H, a]$ . This holds for any choice of  $h_1, h_2 \in U_1$ . In particular, taking  $h_1 = 1$  we see that  $[H, [d, h_2]^{h_2^{-1}}] \leq [H, a]$  while taking  $h_2 = 1$  we conclude that  $[H, [h_1, e]^d] \leq [H, a]$ . It now follows that  $[H, [h_1, h_2]^{dh_2^{-1}}] \leq [H, a]$ . Since  $[H, a]$  is normal in  $H$ , we have  $[H, [h_1, h_2]] \leq [H, a]^{d^{-1}}$  and so  $[H, U_1] \leq [H, a]^{d^{-1}}$ , which proves that  $U_1$  has property 2. Examine again the inclusion  $[H, [d, h_2]^{h_2^{-1}}] \leq [H, a]$ . Since  $[H, a]$  is normal in  $H$ , it follows that  $[H, [U_1, d]] \leq [H, a]$ . Therefore  $U_1$  has property 3 as well. The proof is now complete.  $\square$

We are ready to prove our main results.

**PROOF OF THEOREM 1.1.** Recall that  $G$  is a group in which  $|x^G| \leq n$  for any commutator  $x$ . We need to show that  $|G''|$  is finite and  $n$ -bounded.

We denote by  $X$  the set of commutators in  $G$  and set  $H = G'$ . Let  $m$  be the maximum of indices of  $C_H(x)$  in  $H$ , where  $x \in X$ . Of course,  $m \leq n$ . Choose  $a \in X$  such that  $C_H(a)$  has index precisely  $m$  in  $H$ . Choose  $b_1, \dots, b_m \in H$  such that  $l(b_i) \leq m - 1$  and  $a^H = \{a^{b_i}; i = 1, \dots, m\}$ . Set  $U = C_G(\langle b_1, \dots, b_m \rangle)$ . Note that the index of  $U$  in  $G$  is

$n$ -bounded. Applying Proposition 2.5 with  $K = G$  we find a subgroup  $U_1$ , of  $n$ -bounded index, such that  $[H, U_1'] \leq \langle [H, a]^G \rangle$ . Since the index of  $U_1$  in  $G$  is  $n$ -bounded, we can find  $n$ -boundedly many commutators  $c_1, \dots, c_s \in X$  such that  $H = \langle c_1, \dots, c_s, H \cap U_1 \rangle$ . Let  $T$  be the normal closure in  $G$  of the product of the subgroups  $[H, a]$  and  $[H, c_i]$  for  $i = 1, \dots, s$ . By Lemma 2.3 each of these subgroups has  $n$ -bounded order. Our hypothesis is that each of them has at most  $n$  conjugates. Thus,  $T$  is a product of  $n$ -boundedly many finite subgroups, normalizing each other and having  $n$ -bounded order. We conclude that  $T$  has finite  $n$ -bounded order. Therefore it is sufficient to show that the second derived group of the quotient  $G/T$  has finite  $n$ -bounded order. So we pass to the quotient  $G/T$ . To avoid complicated notation the images of  $G$ ,  $H$  and  $X$  will be denoted by the same symbols. We observe that the derived group of  $HU_1$  is contained in  $Z(H)$ . This follows from the facts that  $HU_1$  is generated by  $c_1, \dots, c_s$  and  $U_1$  and modulo  $T$  we have  $c_1, \dots, c_s \in Z(H)$  and  $U_1' \leq Z(H)$ .

Let  $\mathcal{X}$  denote the family of subgroups  $S \leq G$  with the following properties.

- (1)  $H \leq S$ ;
- (2)  $S' \leq Z(H)$ ;
- (3)  $S$  has finite index in  $G$ .

We already know that  $\mathcal{X}$  is non-empty since it contains  $HU_1$ . Choose  $J \in \mathcal{X}$  of minimal possible index  $j$  in  $G$ . Since the index of  $U_1$  in  $G$  is  $n$ -bounded, the index  $j$  is  $n$ -bounded, too. We will now use induction on  $j$ . If  $j = 1$ , then  $J = G$  and  $H \leq Z(H)$ . So  $G'' = 1$  and we have nothing to prove. Thus, we assume that  $j \geq 2$ .

Again, we take a commutator  $a_0 \in X$  such that  $C_H(a_0)$  has maximal possible index in  $H$  and write  $a_0 = [d, e]$  for suitable  $d, e \in G$ . If both  $d$  and  $e$  belong to  $J$ , we conclude (since  $J' \leq Z(H)$ ) that  $H$  is abelian and  $G'' = 1$ . Thus, assume that at least one of them, say  $d$ , is not in  $J$ . We will use Proposition 2.5 with  $K = G$ . It follows that there is a subgroup  $V$  of  $n$ -bounded index in  $G$  such that  $[H, [V, d]] \leq [H, a_0]$ . Replacing if necessary  $V$  by  $V \cap J$ , without loss of generality we can assume that  $V \leq J$ . Let  $L = J\langle d \rangle$ . Note that  $L' = J'[J, d]$ . Let  $1 = g_1, \dots, g_t$  be a full system of representatives of the right cosets of  $V$  in  $J$ . Then  $[J, d]$  is generated by  $[V, d]^{g_1}, \dots, [V, d]^{g_t}$  and  $[g_1, d], \dots, [g_t, d]$ . This is straightforward from the fact that  $[vg, d] = [v, d]^g[g, d]$  for any  $g, v \in G$ . Next, for each  $i = 1, \dots, t$  set  $x_i = [g_i, d]$ . Let  $R$  be the normal closure in  $G$  of the product of the subgroups  $[H, a_0]^{g_i}$  and  $[H, x_i]$  for  $i = 1, \dots, t$ . By Lemma 2.3 each of these subgroups has  $n$ -bounded order. Our hypothesis is that each of them has at most  $n$  conjugates. Thus,  $R$

is a product of  $n$ -boundedly many finite subgroups, normalizing each other and having  $n$ -bounded order. We conclude that  $R$  has finite  $n$ -bounded order. We see that  $[H, L'] \leq R$ . Since  $d \notin J$ , the index of  $L$  in  $G$  is strictly smaller than  $j$ . Therefore, by induction on  $j$ , the second derived group of  $G/R$  is finite with bounded order. Taking into account that also  $R$  is finite with bounded order, we deduce that  $G''$  is finite with bounded order. The proof is now complete.  $\square$

PROOF OF THEOREM 1.2. Recall that  $G$  is a group in which  $|x^{G'}| \leq n$  for any commutator  $x$ . We need to prove that  $\gamma_3(G')$  is finite with  $n$ -bounded order. As before, we write  $X$  for the set of commutators in  $G$  and  $H$  for the derived group. Choose a commutator  $a \in X$  such that  $C_H(a)$  has maximal possible index in  $H$ . We will use Proposition 2.5 with  $K = H$ . It follows that  $H$  contains a subgroup  $U_1$  of finite  $n$ -bounded index such that  $[H, U_1'] \leq [H, a]^{d^{-1}}$  for some  $d \in G$ . Write  $b_0 = a^{d^{-1}}$  and so  $[H, U_1'] \leq [H, b_0]$ . Since the index of  $U_1$  in  $H$  is  $n$ -bounded, we can find  $n$ -boundedly many commutators  $b_1, \dots, b_s \in X$  such that  $H = \langle b_1, \dots, b_s, U_1 \rangle$ . Let  $T$  be the product of the subgroups  $[H, b_i]$  for  $i = 0, 1, \dots, s$ . By Lemma 2.3 each of these subgroups has  $n$ -bounded order. All of them are normal in  $H$  and so  $T$  is normal in  $H$  and has finite  $n$ -bounded order. The center of  $H/T$  contains the images of  $U_1'$  and  $b_1, \dots, b_s$ . It follows that the quotient of  $H/T$  over its center is abelian. Therefore  $\gamma_3(H) \leq T$ , which completes the proof.  $\square$

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